4 Magnetostatics

Charges give rise to electric fields. Current give rise to magnetic fields. In this section, we will study the magnetic fields induced by steady currents. This means that we are again looking for time independent solutions to the Maxwell equations. We will also restrict to situations in which the charge density vanishes, so $\rho = 0$. We can then set $\mathbf{E} = 0$ and focus our attention only on the magnetic field. We're left with two Maxwell equations to solve:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{4.1}$$

and

$$\nabla \cdot \mathbf{B} = 0 \tag{4.2}$$

If you fix the current density $\mathbf{J}$, these equations have a unique solution. Our goal in this section is to find it.

4.1 Steady current

Before we solve (4.1) and (4.2) let's pause to think about the kind of currents that we're considering in this section. Because $\rho = 0$, there can't be any net charge. But, of course, we still want charge to be moving! This means that we necessarily have both positive and negative charges which balance out at all points in space. Nonetheless, these charges can move so there is a current even though there is no net charge transport.

This may sound artificial, but in fact it's exactly what happens in a typical wire. In that case, there is background of positive charge due to the lattice of ions in the metal.

Meanwhile, the electrons are free to move. But they all move together so that at each point we still have $\rho = 0$. The continuity equation, which captures the conservation of electric charge, is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{4.3}$$

Since the charge density is unchanging (and, indeed, vanishing), we have

$$\nabla \cdot \mathbf{J} = 0 \tag{4.4}$$

Mathematically, this is just saying that if a current flows into some region of space, an equal current must flow out to avoid the build up of charge. Note that this is consistent with (4.1) since, for any vector field, $\nabla \cdot (\nabla \times \mathbf{B}) = 0$

4.2 Force on a moving charge

The most basic form of the magnetic force is the force on a charge q moving with velocity $\mathbf{v}$, exerted by a magnetic field $\mathbf{B}$, which is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{4.5}$$

The SI unit of magnetic field is the tesla (T). One tesla is defined as one newton per ampere-meter. By (4.5) the force on 1 C moving 1 m/s perpendicular to a field of 1 T is 1 N . A magnetic field of 1 T is a rather strong field. The largest fields that can be produced by conventional electromagnets are about 2 T. For high-field research in laboratories, fields of $10-12$ T , produced with superconducting mag- nets, are used. The record for steady-state fields is about 30 T . At the surface of the Earth the magnetic field is about 0.5×10^{-4}T . Very large and very small fields are important in astronomy; e. g.10^{8}T near pulsars, and 10^{-10}T in interstellar space in the galaxy.

The direction of the magnetic force on q is sideways, i.e., perpendicular to $\mathbf{v}$ and to $\mathbf{B}$, as shown in Fig. 8.1 for a positive charge. Therefore, the magnetic force does no work on q :

$$dW = \mathbf{F} \cdot d\mathbf{x} = \mathbf{F} \cdot \mathbf{v} dt = 0 \tag{4.6}$$

The magnetic force affects the direction of motion of q, but not its kinetic energy. Because the magnetic force does no work, it is not possible to define a potential energy function for the magnetic force. The magnetic force is velocity dependent, which is quite different from the other fundamental forces we encounter in physics.

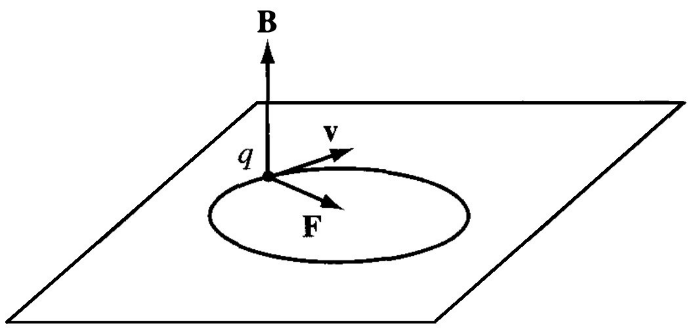

Figure 4.1: The magnetic force. The force on a moving charge is sideways, perpendicular to both v and B. The figure shows F and the circular trajectory for a positive charge in a uniform field.

Equation (4.5) may be taken as the definition of the magnetic field. From the measurement of the force on a test charge q, e.g. by observing its deflection for a known velocity, the defined quantity $\mathbf{B}$ could in principle be deduced from (4.5)

When both electric and magnetic field is present then the force equation is known as lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{4.7}$$

4.2.1 Force on current carrying wire

A current-carrying wire in a magnetic field experiences a force, from the magnetic force on the individual moving charges that make the current. Electric current is motion of charge, and (4.5) is the force on a moving charge. Suppose the current consists of particles with charge q and linear density n_1 , moving with mean velocity $\mathbf{v}$. Then the net force $d\mathbf{F}$ on a small segment $d\ell$ of the wire is

$$d\mathbf{F} = (n_1 d\ell)\, q\mathbf{v} \times \mathbf{B} \tag{4.8}$$

because $n_1 d\ell$ is the number of moving charges in $d\ell$. The current I in the wire is $qn_1 v$, and v is parallel to $d\ell$, so the total force on the wire is

$$\mathbf{F} = \int_{\text{wire}} I d\ell \times \mathbf{B} \tag{4.9}$$

4.3 Electric current as a source of magnetic field

Where does a magnetic field come from? What is its source? The most familiar source is a permanent magnet, i.e., a piece of magnetized iron or other ferromagnetic material. But at a more basic level the source of $\mathbf{B}$ is to be found in electric current. We will study magnetic materials, and see that the magnetic and field of a ferromagnet comes from properties of atomic electrons - their spin and orbital motion - which, although not classical currents, do involve dynamics of charged particles. However, now we are concerned with $\mathbf{B}$ from macroscopic steady currents.

4.3.1 Biot-Savart Law

The magnetic field of a steady line current is given by the Biot-Savart law: The field $d\mathbf{B}$ at a point P, due to an infinitesimal current element $Id\ell$ at a point P'. Figure below shows the geometry. For this elemental case the Biot-Savart law is

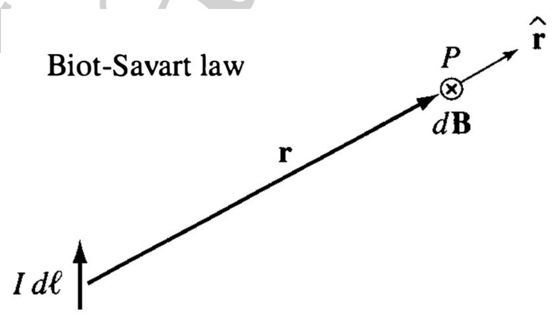

Figure 4.2: Elemental form of the Biot-Savart law. $Id\ell$ is the source of magnetic field, and $d\mathbf{B}$ is the resulting field at P .

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\ell \times \hat{\mathbf{r}}}{r^2} \tag{4.10}$$

Here r is the distance between P' (the source point) and P (the field point), $\hat{\mathbf{r}}$ is the unit vector in the direction from P' to P, and $\mathbf{r} = r\hat{\mathbf{r}}$. Note that the Biot-Savart law is an inverse-square law, like Coulomb's law, but the direction of the magnetic field is azimuthal, around the axis of $Id\ell$.

Do few examples from Griffiths.

4.3.2 Ampere's Law

The integral of the current density over the surface S is the same thing as the total current I that passes through S. Ampère's law in integral form then reads

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I \tag{4.11}$$

For the problems where cylindrical symmetry is there, it is easy to apply Ampere's law to calculate Magnetic field.

Consider an infinite, straight wire carrying current I. We'll take it to point in the $\hat{\mathbf{z}}$ direction. The symmetry of the problem is jumping up and down telling us that we need to use cylindrical polar coordinates, (r, φ, z), where $r = \sqrt{x^2 + y^2}$ is the radial distance away from the wire.

We take the open surface S to lie in the $x - y$ plane, centered on the wire. For the line integral in (4.11) to give something that doesn't vanish, it's clear that the magnetic field has to have some component that lies along the circumference of the disc.

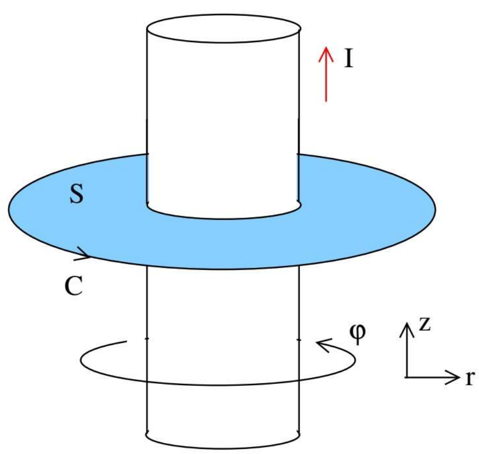

Figure 4.3: Magnetic field of long straight wire

But, by the symmetry of the problem, that's actually the only component that B can have: it must be of the form $B = B(r)\hat{\varphi}$. (If this was a bit too quick, we'll derive this more carefully below). Any magnetic field of this form automatically satisfies the second Maxwell equation $\nabla \cdot \mathbf{B} = 0$. We need only worry about Ampère's law which tells us

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = B(r) \int_0^{2\pi} r d\varphi = 2\pi r B(r) = \mu_0 I$$

We see that the strength of the magnetic field is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi} \tag{4.12}$$

The magnetic field circles the wire using the “right-hand rule”: stick the thumb of your right hand in the direction of the current and your fingers curl in the direction of the magnetic field.

4.4 Surface Currents and Discontinuities

Consider the flat plane lying at $z = 0$ with a surface current density that we'll call $\mathbf{K}$. Note that $\mathbf{K}$ is the current per unit length, as opposed to $\mathbf{J}$ which is the current per unit area. You can think of the surface current as a bunch of wires, all lying parallel to each other.

We'll take the current to lie in the x-direction: $\mathbf{K} = K\hat{\mathbf{x}}$ as shown below.

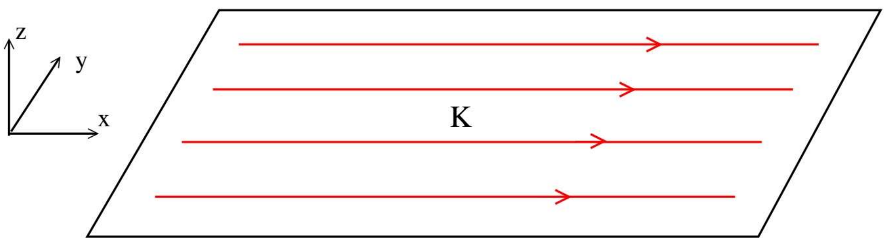

Figure 4.4: Surface Current

From our previous result, we know that the B field should curl around the current in the right-handed sense. But, with an infinite number of wires, this can only mean that B is oriented along the y direction. In fact, from the symmetry of the problem, it must look like

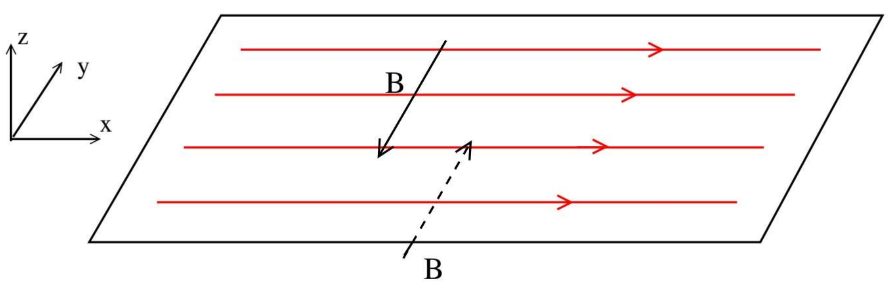

Figure 4.5: Magnetic field due to a Surface Current

with $\mathbf{B}$ pointing in the $-\hat{\mathbf{y}}$ direction when $z > 0$ and in the $+\hat{\mathbf{y}}$ direction when $z < 0$ we write

$$\mathbf{B} = -B(z)\hat{\mathbf{y}}$$

with $B(z) = -B(-z)$. We invoke Ampère's law using the following open surface:

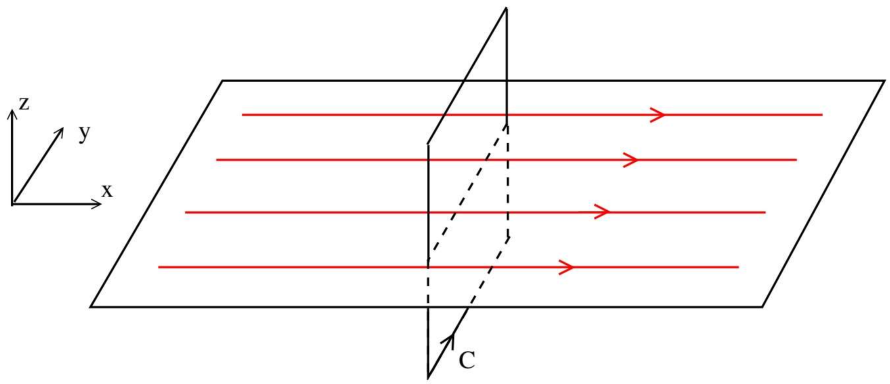

Figure 4.6:

with length L in the y direction and extending to $\pm z$. We have

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = LB(z) - LB(-z) = 2LB(z) = \mu_0 K L$$

so we find that the magnetic field is constant above an infinite plane of surface current

$$B(z) = \frac{\mu_0 K}{2} \quad z > 0$$

This is rather similar to the case of the electric field in the presence of an infinite plane of surface charge.

The analogy with electrostatics continues. The magnetic field is not continuous across a plane of surface current. We have

$$B\left(z \rightarrow 0^{+}\right) - B\left(z \rightarrow 0^{-}\right) = \mu_0 K$$

In fact, this is a general result that holds for any surface current $\mathbf{K}$. We can prove this statement by using the same curve that we used in the Figure above and shrinking it until it barely touches the surface on both sides. If the normal to the surface is $\hat{\mathbf{n}}$ and $\mathbf{B}_{\pm}$ denotes the magnetic field on either side of the surface, then

$$\hat{\mathbf{n}} \times \mathbf{B}|_{+} - \hat{\mathbf{n}} \times \mathbf{B}|_{-} = \mu_0 \mathbf{K} \tag{4.13}$$

Meanwhile, the magnetic field normal to the surface is continuous. We can write the boundary conditions as

$$B_{\text{above}}^{\perp} = B_{\text{below}}^{\perp} \tag{4.14}$$

and

$$B_{\text{above}}^{\|} - B_{\text{below}}^{\|} = \mu_0 K \tag{4.15}$$

When we looked at electric fields, we saw that the normal component was discontinuous in the presence of surface charge while the tangential component is continuous.

For magnetic fields, it's the other way around: the tangential component is discontinuous in the presence of surface currents.

4.5 Magnetic Vector Potential

At this point it is useful to compare the principles of magnetostatics and electrostatics, given in the table below.

Electrostatics	Magnetostatics
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \times \mathbf{E} = 0$
$\mathbf{B}$ curls around I	$\mathbf{E}$ diverges from q

Note this difference: The electric field has scalar sources, but the magnetic field has vector sources. Electric charge, i.e., a point source of $\mathbf{E}$ from which $\mathbf{E}$ diverges, is a common property of elementary particles. Motion of charged particles-current-is the source of $\mathbf{B}$, around which $\mathbf{B}$ curls.

Magnetic charge, i.e., a point source from which B diverges in the rest frame of the particle, apparently does not exist in nature, or at least it has not been observed. Such a hypothetical charge is called a magnetic monopole. Dirac showed that it is possible to construct a consistent quantum theory with both electric charges and magnetic monopoles. However, the fundamental magnetic charge g and electric charge e would necessarily be quantized, and satisfy the relation $eg = n/2$, where n is an integer. Many experimenters have searched for magnetic monopoles, but so far the results are negative. Some speCUlative theories of high-energy physics, such as grand unified field theories, predict the existence of very massive magnetic monopoles, too massive to be produced at current high-energy accelerators,

but which might have been produced in the big bang. Searching for magnetic monopoles continues to be an interesting experimental challenge

If magnetic monopoles do not exist, then the equation $\nabla \cdot \mathbf{B} = 0$ is a universal equation of magnetism. Whether magnetic monopoles exist or not, the source of the magnetic fields we encounter in physics are not point magnetic charges but rather currents of electric charge, corresponding to the source equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$.

We found in electrostatics that it is useful to introduce a scalar potential $V(\mathbf{x})$ for the electrostatic field, such that $\mathbf{E} = -\nabla V$. This guarantees that $\nabla \times \mathbf{E} = 0$. In an analogous way we may introduce a vector potential $\mathbf{A}(\mathbf{x})$ for the magnetic field, such that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.16}$$

This guarantees that $\nabla \cdot \mathbf{B} = 0$

However, (4.16) does not uniquely determine $\mathbf{A}$ for a given magnetic field $\mathbf{B}$: If $f(\mathbf{x})$ is an arbitrary scalar function, then $\mathbf{A} + \nabla f$ has the same curl as $\mathbf{A}$ (namely B) because $\nabla \times \nabla f$ is identically 0. Therefore we may impose a condition on $\mathbf{A}$ called a gauge condition, to remove this ambiguity. The

Coulomb gauge condition, which we will use in magnetostatics, is

$$\nabla \cdot \mathbf{A} = 0 \tag{4.17}$$

Taking (4.16) and (4.17) together is still not enough to make $\mathbf{A}(\mathbf{x})$ unique, because adding a constant does not change either $\nabla\times\mathbf{A}$ or $\nabla\cdot\mathbf{A}$. But imposing an appropriate boundary condition, such as requiring $\mathbf{A} \to 0$ at infinity, makes $\mathbf{A}$ unique.

Example: Consider the uniform field $\mathbf{B} = B_0\mathbf{k}$. A vector potential function, satisfying (4.16) for the uniform field and the Coulomb gauge condition (4.17) is

$$\mathbf{A}(\mathbf{x}) = \frac{1}{2}\mathbf{B} \times \mathbf{x} = \frac{1}{2}B_0(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})$$

An example of uniform $\mathbf{B}$ is the field inside a solenoid, so for $\mathbf{B} = B_0\hat{\mathbf{k}}$ we may picture a long, tightly and uniformly wound solenoid, whose axis is the z axis. Note for this case that $\mathbf{A}(\mathbf{x})$ is parallel to the surface currents (azimuthal) and that $\mathbf{A}(\mathbf{x})$ curls around the $\mathbf{B}$ field.

The analytic expression of vector potential in vector notation is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{4.18}$$

4.6 Magnetic Dipole

Vector potential, while in the general coordinate as shown below figure, can be expanded as

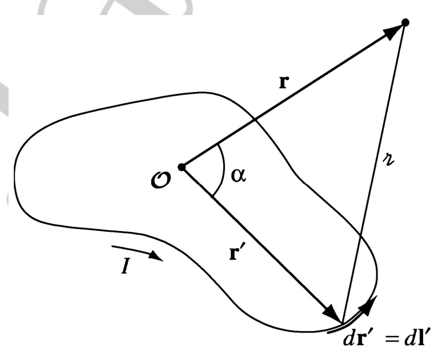

Figure 4.7: Multipole expansion of vector potential

As in the multipole expansion of V, we call the first term (which goes like $1/r$) the monopole term, the second (which goes like $1/r^2$) dipole, the third quadrupole, and so on.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\alpha d\mathbf{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)d\mathbf{l}' + \cdots\right] \tag{4.19}$$

The magnetic monopole term is always zero

The dipole term

$$\mathbf{A}_{\text{dip}}(\mathbf{r})=\frac{\mu_0 I}{4\pi r^2}\oint r'\cos\alpha d\mathbf{l}'=\frac{\mu_0 I}{4\pi r^2}\oint(\hat{\mathbf{r}}\cdot\mathbf{r}')\,d\mathbf{l}' \tag{4.20}$$

We define the dipole moment

$$\mathbf{m}\equiv I\int d\mathbf{a}=I\mathbf{a} \tag{4.21}$$

Then the vector potential for the the dipole term

$$\mathbf{A}_{\text{dip}}(\mathbf{r})=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \tag{4.22}$$

Magnetic field of a magnetic dipole can be written as

$$\mathbf{B}_{\text{dip}}(\mathbf{r})=\frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}] \tag{4.23}$$

The magnetic field of a (perfect) dipole is easiest to calculate if we put $\mathbf{m}$ at the origin and let it point in the z -direction see figure below.

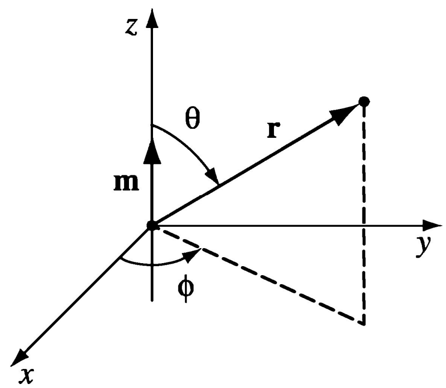

Figure 4.8: Calculation of magnetic field due to a magnetic dipole

According to Equation $\mathbf{A}_{\text{dip}}(\mathbf{r})=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$ the potential at point (r,θ,ϕ) is

$$\mathbf{A}_{\text{dip}}(\mathbf{r})=\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} \tag{4.24}$$

Hence the magnetic field

$$\mathbf{B}_{\text{dip}}(\mathbf{r})=\nabla\times\mathbf{A}=\frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}) \tag{4.25}$$

Example: Find the magnetic dipole moment of the book type loop shown in Figure below. All sides have length w, and it carries a current I.

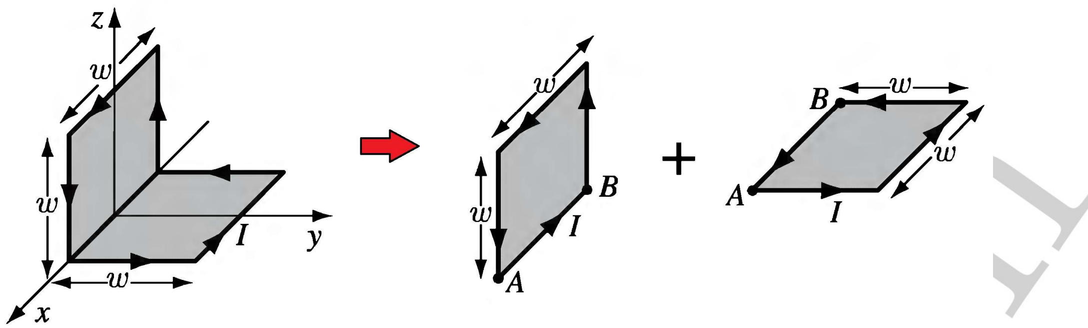

Figure 4.9:

Solution: This wire could be considered the superposition of two plane square loops The "extra" sides (AB) cancel when the two are put together, since the currents flow in opposite directions. The net magnetic dipole moment is

$$\mathbf{m} = Iw^2\hat{\mathbf{y}} + Iw^2\hat{\mathbf{z}}$$

its magnitude is $\sqrt{2}Iw^2$, and it points along the 45° line $z = y$

Example: A circular loop of wire, with radius R, lies in the xy plane (centered at the origin) and carries a current I running counterclockwise as viewed from the positive z axis.

(a) What is its magnetic dipole moment?
(b) What is the (approximate) magnetic field at points far from the origin?
(c) Show that, for points on the z axis, your answer is consistent with the exact field $\mathbf{B} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$ when $z \gg R$.

Solution: (a) $\mathbf{m} = I\mathbf{a} = I\pi R^2\hat{\mathbf{z}}$

(b) $\mathbf{B} \approxeq \frac{\mu_0}{4\pi}\frac{I\pi R^2}{r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})$

(c) On the z axis, $\theta = 0, r = z, \hat{\mathbf{r}} = \hat{\mathbf{z}}($ for $z > 0)$. So

$$\mathbf{B} \approxeq \mathbf{B} \approx \frac{\mu_0 I R^2}{2z^3}\hat{\mathbf{z}}$$

for $z < 0, \theta = \pi, \hat{\mathbf{r}} = -\hat{\mathbf{z}}$, so the field is the same, with $|z|^3$ in place of z^3

The original ans $\left(\mathbf{B} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\right)$ reduces (for $z \gg R$) to $B \approx \mu_0 I R^2/2|z|^3$. So the ans are same.

Example: (a) A disk of radius R, carrying a uniform surface charge σ, is rotating at constant angular velocity ω . Find its magnetic dipole moment.

(b) A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find the magnetic dipole moment of the spinning spherical shell.

Also find the magnetic vector potential of the spinning shell.

Solution: (a) For a ring, $m = I\pi r^2$. So the current due to a ring with radius extended to small amount $r \to r + dr$ is (see figure (a) below)

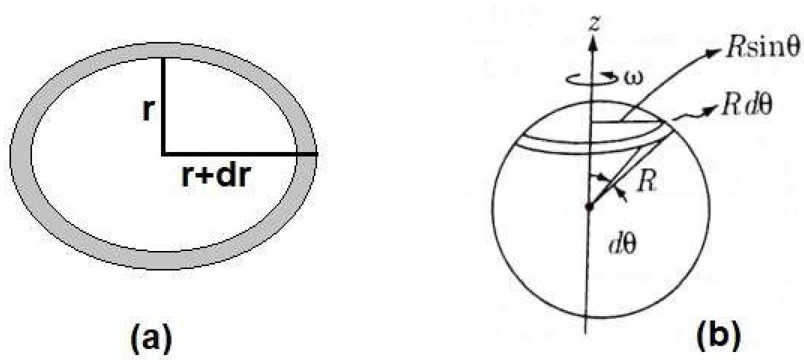

Figure 4.10:

$$I \to \sigma v dr = \sigma\omega r dr$$

So the magnetic dipole moment

$$m = \int_0^R \pi r^2 \sigma\omega r dr = \pi\sigma\omega R^4/4$$

(b) See figure (b) above. The total charge on the shaded ring is $dq = \sigma(2\pi R\sin\theta)R d\theta$. The time for one revolution is $dt = 2\pi/\omega$.

So the current in the ring is $I = \frac{dq}{dt} = \sigma\omega R^2 \sin\theta d\theta$. The area of the ring is $\pi(R\sin\theta)^2$, so the magnetic moment of the ring is $dm = (\sigma\omega R^2 \sin\theta d\theta)\,\pi R^2 \sin^2\theta$, and the total dipole moment of the shell is

$$m = \sigma\omega\pi R^4 \int_0^\pi \sin^3\theta d\theta = (4/3)\sigma\omega\pi R^4$$

Hence

$$\mathbf{m} = \frac{4\pi}{3}\sigma\omega R^4 \hat{\mathbf{z}}$$

Dipole term of the vector potential is

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

In this problem

$$\mathbf{A}_{\text{dip}} = \frac{\mu_0}{4\pi}\frac{4\pi}{3}\sigma\omega R^4 \frac{\sin\theta}{r^2}\hat{\phi} = \frac{\mu_0\sigma\omega R^4}{3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

4.6.1 Torques and Forces on Magnetic Dipoles

Torque on a magnetic dipole of dipole moment m while placed in magnetic field B

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \tag{4.26}$$

Force on a magnetic dipole is zero when placed in a uniform field. But when placed in a non-uniform field then there is a net force acts

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \tag{4.27}$$

Potential energy of a magnetic dipole while placed in a magnetic field is

$$U = -\mathbf{m} \cdot \mathbf{B} \tag{4.28}$$

4.7 Electric Current

The conceptually simplest example of an electric current is the current in a thin conducting wire. In an ideal one-dimensional wire the current I is defined as the net charge passing a point P per unit time. In a real wire I is defined as the charge per unit time passing through a cross section of the wire at P.

$$I = \frac{dQ}{dt} \tag{4.29}$$

The unit of current is the ampere (A), which is the basic electric unit in the SI system. If the wire carries a current of 1 A at point P, then 1 C of net charge passes P each second.

If the current in a wire is due to charges q moving with mean velocity v, and the charges have linear density n_L (= number of charge carriers per unit length) , then

$$I = qn_Lv \tag{4.30}$$

In n is the charge carrier density per unit volume and A is the surface area of the current carrying wire, q is the amount of charge each particle carries, then the current is

$$I = qnAv \tag{4.31}$$

What is the current for an orbiting electron? Well the charge is e, velocity is v and the radius of atom is taken to be R. Then the electron crosses a point once in one time period. So the current is

$$I = \frac{e}{T}$$

As T is $2\pi R/v$, the current

$$I = \frac{ev}{2\pi R} \tag{4.32}$$

You will need this expression many times

4.7.1 Current Density

If the current in a volume of space is due to charges q with volume number density n (= number of charge carriers per unit volume) moving with mean velocity v , then the current density is

$$\mathbf{J} = qn\mathbf{v} \tag{4.33}$$

Note that the units of $qn\mathbf{v}$ are $\mathrm{A/m^2}$. Current density is the flux of electric charge. In general flux is equal to density times velocity.

4.7.2 Conservation of Charge

The net charge of an isolated system is constant. But charge is not only conserved overall in a system, it is also conserved point by point throughout the system. This local conservation of charge is described mathematically by the continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{4.34}$$

Here $\rho(\mathbf{x}, t)$ is the volume charge density (= charge per unit volume), and $\mathbf{J}(\mathbf{x}, t)$ is the volume current density (= current per unit area). Equation (4.34) is universally true, for arbitrary time dependence.

Integral form of the continuity equation,

$$\oint_S \mathbf{J} \cdot d\mathbf{A} = -\frac{d}{dt} \int_V \rho d^3x \tag{4.35}$$

states that the rate of charge passing outward through the closed surface S is equal to the rate of decrease of charge in the enclosed volume V. Equation (4.34), or equivalently (4.35), is a basic equation of electrodynamics, expressing local conservation of charge.

4.7.3 Ohm's Law

How is the current related to the potential gradient in a conductor? If two terminal points on a conductor are held at a constant potential difference V, e.g., by connecting them to the electrodes of a battery, then in equilibrium a steady current flows through the conductor. Let I be the total current at either point, i.e., the integrated flux $\int \mathbf{J} \cdot d\mathbf{A}$ through a surface inside the conductor surrounding the point. It is found empirically that formany cases the current and potential difference are proportional,

$$V = IR \tag{4.36}$$

The constant of proportionality R is called the resistance of the conductor. The SI unit of resistance is the ohm (Ω), defined by $1\Omega = 1\mathrm{V/A}$. The reciprocal $1/R$ is called the conductance, and its unit is Ω^{-1}, or siemens (S) . Ohm's law holds to a very good approximation for many conductors. However, it is not a universal principle, there are examples where it does not hold.

The resistance R of a sample of matter is a function of the geometry (size and shape) of the sample, and of the material composition. For example, the resistance of a uniform cylinder, of length L and cross section A, is proportional to L and inversely proportional to A,

$$R = \rho L/A \tag{4.37}$$

The parameter ρ (not to be confused with charge density $\rho(\mathbf{x})$! is an intrinsic property of the material called the resistivity.

We may also write a local form of Ohm's law, which is a more basic equation.

$$\mathbf{J}(\mathbf{x}) = \sigma \mathbf{E}(\mathbf{x}) \tag{4.38}$$

Again, σ is the conductivity.

Example Two long coaxial metal cylinders (radii a and b) are separated by material of conductivity σ as shown in figure below . If they are maintained at a potential difference V, what current flows from one to the other, in a length L?

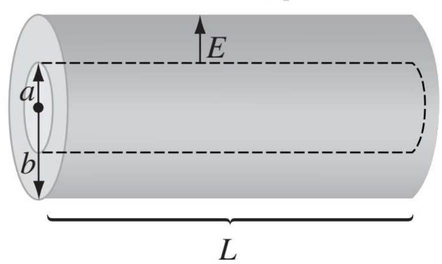

Figure 4.11:

Solution: The field between the cylinders is

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$$

where λ is the charge per unit length on the inner cylinder. The current is

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma \int \mathbf{E}\cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}\lambda L$$

The integral is over any surface enclosing the inner cylinder. The potential difference between the cylinders is

$$V = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

$$I = \frac{2\pi\sigma L}{\ln(b/a)}V$$

So the resistance of the system is

$$R = \frac{\ln(b/a)}{2\pi\sigma L}$$

5 Magnetic Field in a medium

Electric fields are created by charges; magnetic fields are created by currents. We learned in our first course that the simplest way to characterise any localised current distribution is through a magnetic dipole moment $\mathbf{m}$. For example, a current I moving in a planar loop of area A with normal $\hat{\mathbf{n}}$ has magnetic dipole moment,

$$\mathbf{m} = IA\hat{\mathbf{n}}$$

Magnetic vector potential and magnetic field due to a magentic dipole are

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} \Rightarrow \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}\right) \tag{5.1}$$

Current loops, and their associated dipole moments, already exist inside materials. They arise through two mechanisms:

- Electrons orbiting the nucleus carry angular momentum and act as magnetic dipole moments.
- Electrons carry an intrinsic spin. This is purely a quantum mechanical effect. This too contributes to the magnetic dipole moment.

We define the magnetisation $\mathbf{M}$ to be the average magnetic dipole moment per unit volume.

In most (but not all) materials, if there is no applied magnetic field then the different atomic dipoles all point in random directions. This means that, after averaging, $\langle\mathbf{m}\rangle = 0$ when $\mathbf{B} = 0$. However, when a magnetic field is applied, the dipoles line up. The magnetisation typically takes the form $\mathrm{M} \propto \mathrm{B}$. We're going to use a slightly strange notation for the proportionality constant. (It's historical but, as we'll see, it turns out to simplify a later equation)

$$\mathbf{M} = \frac{1}{\mu_0}\frac{\chi_m}{1+\chi_m}\mathbf{B} \tag{5.2}$$

where χ_m is the magnetic susceptibility. The magnetic properties of materials fall into three different categories. The first two are dictated by the sign of χ_m :

- Diamagnetism: $-1 < \chi_m < 0$. The magnetisation of diamagnetic materials points in the opposite direction to the applied magnetic field. Most metals are diamagnetic, including copper and gold. Most non-metallic materials are also diamagnetic, including, importantly, water with $\chi_m \approx -10^{-5}$. This means, famously, that frogs are also diamagnetic. Superconductors can be thought of as "perfect" diamagnets with $\chi_m = -1$.
- Paramagnetism: $\chi_m > 0$. In paramagnets, the magnetisation points in the same direction as the field. There are a number of paramagnetic metals, including Tungsten, Cesium and Aluminium.
- Ferromagnetism: $\mathrm{M} \neq 0$ when $\mathrm{B} = 0$. Materials with this property are what you usually call "magnets". They're the things stuck to your fridge. The direction of B is from the south pole to the north. Only a few elements are ferromagnetic. The most familiar is Iron. Nickel and Cobalt are other examples.

5.1 Bound Currents

When a material becomes magnetised (at least in an anisotropic way), there will necessarily be regions in which there is a current. This is called the bound current.

Let's first give an intuitive picture for where these bound currents appear from. Consider a bunch of equal magnetic dipoles arranged uniformly on a plane like the left picture : The currents in the interior region cancel out and we're left only with a surface current around the edge. We know that this is surface current a $\mathbf{K}$. We'll follow this notation and call the surface current arising from a constant, internal magnetisation $\mathbf{K}_{\text{bound}}$.

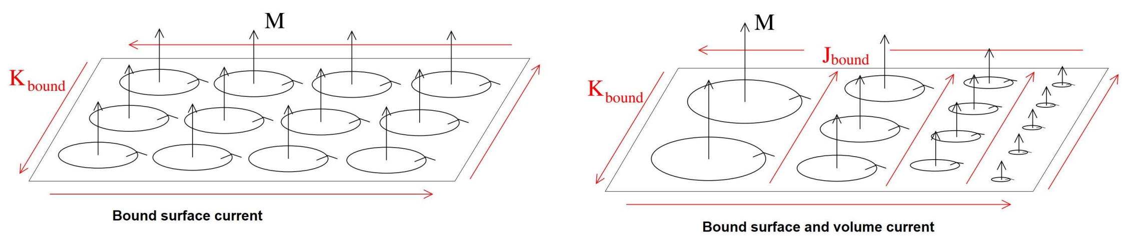

Figure 5.1: Origin of bound surface and volume current

Now consider instead a situation where the dipoles are arranged on a plane, but have different sizes. We'll put the big ones to the left and the small ones to the right, like In this case, the currents in the interior no longer cancel. As we can see from the right side of the picture picture, they go into the page. since M is out of the page, and we've arranged things so that M varies from left to right, this suggests that $\mathbf{J}_{\text{bound}} \sim \nabla \times \mathbf{M}$.

Bound surface and volume currents are

$$\mathbf{K}_{\text{bound}} = \mathbf{M} \times \hat{\mathbf{n}} \tag{5.3}$$

and

$$\mathbf{J}_{\text{bound}} = \nabla \times \mathbf{M} \tag{5.4}$$

Note that the bound current is a steady current, in the sense that it obeys $\nabla \cdot \mathbf{J}_{\text{bound}} = 0$.

5.2 Magnetic intensity

Recall that Amp'ere's law describes the magnetic field generated by static currents. We've now learned that, in a material, there can be two contributions to a current: the bound current $\mathbf{J}_{\text{bound}}$ that we've discussed above, and the current $\mathbf{J}_{\text{free}}$ from freely flowing electrons that we were implicitly talking. Amp'ere's law does not distinguish between these two currents; the magnetic field receives contributions from both.

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0 (\mathbf{J}_{\text{free}} + \mathbf{J}_{\text{bound}}) \\ &= \mu_0 \mathbf{J}_{\text{free}} + \mu_0 \nabla \times \mathbf{M} \end{aligned} \tag{5.5}$$

We define the Magnetic Intensity or magnetising field, $\mathbf{H}$ as

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{5.6}$$

This obeys

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{ free}} \tag{5.7}$$

and in integral form

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{f}_{\text{enc}}} \tag{5.8}$$

We see that the field $\mathbf{H}$ plays a similar role to the electric displacement $\mathbf{D}$; the effect of the bound currents have been absorbed into $\mathbf{H}$, so that only the free currents contribute. Note, however, that we can't quite forget about $\mathbf{B}$ entirely, since it obeys $\nabla \cdot \mathbf{B} = 0$. In contrast, we don't necessarily have "$\nabla \cdot \mathbf{H} = 0$''. Rather annoyingly, in a number of books $\mathbf{H}$ is called the magnetic field and $\mathbf{B}$ is called the magnetic induction. But this is stupid terminology so we won't use it.

5.2.1 Linear magnetic material

For most of the magnetic material the magnetization vector and magnetic intensity are proportional. They are called linear magnetic material

$$\mathbf{M} = \chi_m \mathbf{H} \tag{5.9}$$

The constant of proportionality χ_m is called the magnetic susceptibility; it is a dimensionless quantity that varies from one substance to another - positive for paramagnets and negative for diamagnets.

From the definition of $\mathbf{H}$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0 \left(1 + \chi_m\right) \mathbf{H}$$

or

$$\mathbf{B} = \mu \mathbf{H} \tag{5.10}$$

where

$$\mu \equiv \mu_0 \left(1 + \chi_m\right) \tag{5.11}$$

μ is called the permeability of the material.

Do some example from Griffiths.

5.3 Electromagnetic Boundary conditions

5.3.1 Magnetostatic Boundary condition inside a medium

Boundary conditions inside the magnetic medium.

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0 \tag{5.12}$$

and

$$\mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}} \tag{5.13}$$

this can also be written as in terms of B

$$\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \tag{5.14}$$

5.3.2 Boundary condition in both electric and magnetic field

In the presence of surface charge, the electric field normal to the surface is discontinuous, while the electric field tangent to the surface is continuous. For magnetic fields, it's the other way around: in the presence of a surface current, the magnetic field normal to the surface is continuous while the magnetic field tangent to the surface is discontinuous.

What happens with dielectrics? Now we have two options of the electric field, $\mathbf{E}$ and $\mathbf{D}$, and two options for the magnetic field, $\mathbf{B}$ and $\mathbf{H}$. They can't both be continuous because they're related by $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ and we'll be interested in situation where $\epsilon($ and possibly $\mu)$ are different on either side. Nonetheless, we can use the same kind of computations that we saw previously to derive the boundary conditions. Roughly, we get one boundary condition from each of the Maxwell equations.

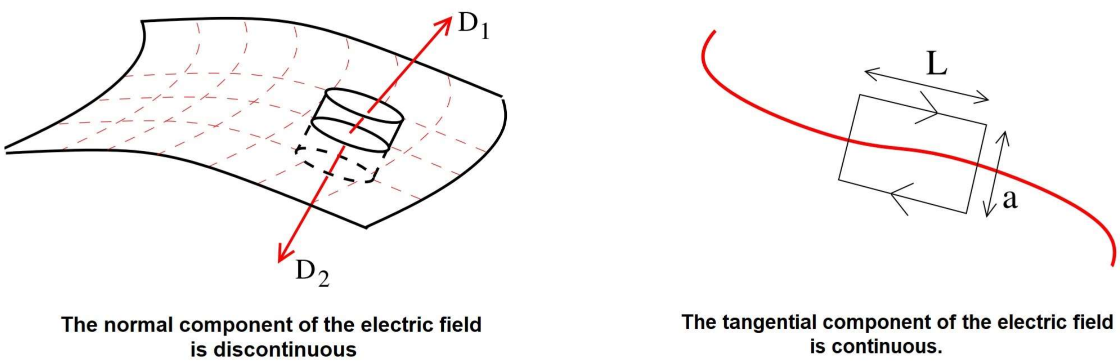

Figure 5.2: Boundary conditions of electric field

For example, consider the Gaussian pillbox shown in the left-hand figure above. Integrating the Maxwell equation $\nabla \cdot \mathbf{D} = \rho_{\text{ free}}$ tells us that the normal component of $\mathbf{D}$ is discontinuous in the presence of surface charge,

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma \tag{5.15}$$

where $\hat{\mathbf{n}}$ is the normal component pointing from 1 into 2. Here σ refers only to the free surface charge. It does not include any bound charges. Similarly, integrating $\nabla \cdot \mathbf{B} = 0$ over the same Gaussian pillbox tells us that the normal component of the magnetic field is continuous.

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \tag{5.16}$$

To determine the tangential components, we integrate the appropriate field around the loop shown in the right-hand figure above. By Stoke's theorem, this is going to be equal to the integral of the curl of the field over the bounding surface. This tells us what the appropriate field is: it's whatever appears in the Maxwell equations with a curl. So if we integrate $\mathbf{E}$ around the loop, we get the result

$$\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \tag{5.17}$$

Meanwhile, integrating $\mathbf{H}$ around the loop tells us the discontinuity condition for the magnetic field

$$\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K} \tag{5.18}$$

where $\mathbf{K}$ is the surface current.

5.3.3 Example: Field inside the gap

Here is a very interesting problem. A C-magnet is shown in Figure . All dimensions are in cm. The relative permeability of the soft Fe yoke is 3000 . If a current $I = 1$ amp is to produce a field of about 100 gauss in the gap, how many turns of wire are required?

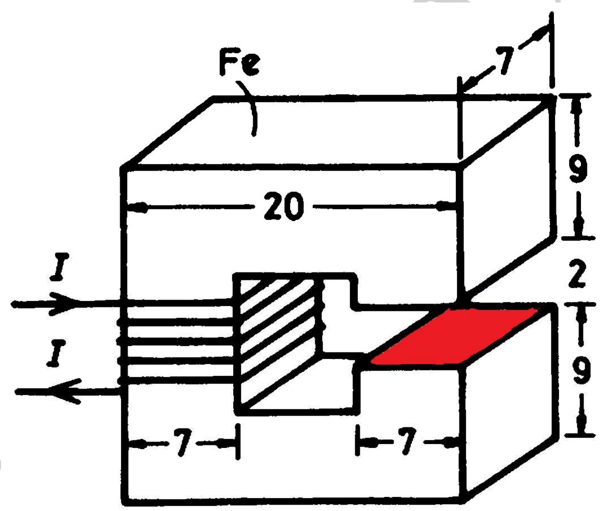

Figure 5.3: Magnetic field in the gap

Solution: You must know the boundary condition of the magnetic field. Consider the surfaces which act like the cross section of the magnet (red colored in the image). The normal component of $\mathbf{B}$ is continuous

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0$$

And there are no parallel component of Magnetic field (remember that in a solenoid the field is along the axis only). So, the magnetic intensity in the gap is $H = \frac{B}{\mu_0}$, while that inside the magnet is $H = \frac{B}{\mu_0 \mu_r}$, where μ_r is the relative permeability of the iron. Ampere's circuital law

$$\oint \mathbf{H} \cdot dI = NI$$

applied to the closed loop passing through inside the magnet and covering the whole magnet

$$\frac{B}{\mu_0}d + \frac{B}{\mu_0\mu_r}(4l - d) = NI$$

hence

$$N = \frac{B}{\mu_0 I}\left[d + \frac{1}{\mu_r}(4l - d)\right]$$

now apply the given data - $l = 20$ cm $= 0.2$ m , $d = 2$ cm $= 0.02$ m, and $B = 100$ gauss $= 100 \times 10^{-4}$ Tesla.

$$N = \frac{100 \times 10^{-4}}{4\pi \times 10^{-7} \times 1}\left(0.02 + \frac{0.2 \times 4 - 0.02}{3000}\right)$$
$$= 161 \text{ turns}$$

6 Changing Electric and Magnetic Field

6.1 Motional emf

When charge moves there acts the Lorenz force on them

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{6.1}$$

The charges move according the force experienced by them. The motion of the charges generates a current and thus an emf on the medium in which the charges are moving. This is called motional emf.

Lorentz force law is very useful to determine the direction of the induced current. To calculate the magnitude we need to know the Faraday's law of induction.

6.2 Faraday's Law

When the magnetic flux ($\Phi = \int \mathbf{B} \cdot \mathbf{dS}$) somehow changes in a closed conducting system (be it a closed loop of wire or something a bend and closed rod) there creates an emf

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{6.2}$$

The significance of the negative sign is, the induced emf tries to oppose the change of the magnetic flux.

The basic principle the Faraday's law states is: **A changing magnetic field induces an electric field.**

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{6.3}$$

This is Faraday's law, in integral form. We can convert it to differential form by applying Stokes' theorem:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.4}$$

This equation tells us that if you change a magnetic field, you'll create an electric field. In turn, this electric field can be used to accelerate charges which, in this context, is usually thought of as creating a current in wire. The process of creating a current through changing magnetic fields is called induction.

Faraday's law tells us that if you change the magnetic flux through S then a current will flow. There are a number of ways to change the magnetic field. You could simply move a bar magnet in the presence of circuit, passing it through the surface S; or you could replace the bar magnet with some other current density, restricted to a second wire C', and move that; or you could keep the second wire C' fixed and vary the current in it, perhaps turning it on and off. All of these will induce a current in C.

However, there is then a secondary effect. When a current flows in C, it will create its own magnetic field. This induced magnetic field will always be in the direction that opposes the change. This is called Lenz's law. If you like, "Lenz's law" is really just the minus sign in Faraday's law.

Example A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart (see figure below) . A resistor R is connected across the rails, and a uniform magnetic field $\mathbf{B}$, pointing into the page, fills the entire region.

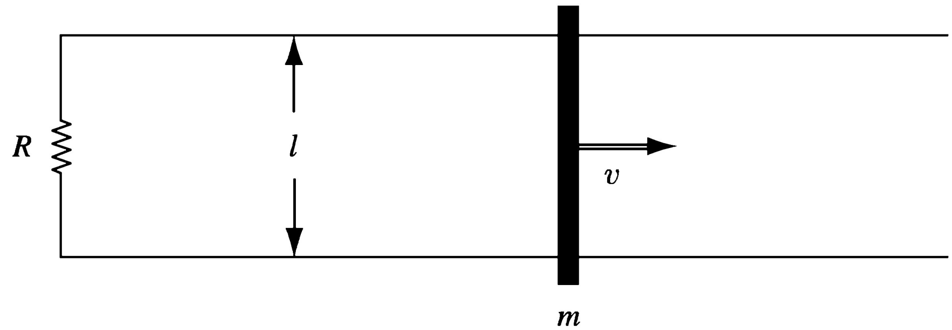

Figure 6.1: Current created by moving bar in a magnetic field

(a) If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

(b) What is the magnetic force on the bar? In what direction?

(c) If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t ?

(d) The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0^2$.

Solution: (a) $\mathcal{E} = -\frac{d\Phi}{dt} = -Bl\frac{dx}{dt} = -Blv; \mathcal{E} = IR \quad \Rightarrow \quad I = \frac{Blv}{R}$. The minus sign tells you the direction of flow of the current. $(\mathbf{v} \times \mathbf{B})$ is upward, in the bar, so the current is downward through the resistor.

(b) $F = IlB = \frac{B^2l^2v}{R}$, to the left.

(c) $F = ma = m\frac{dv}{dt} = -\frac{B^2l^2}{R}v \Rightarrow \frac{dv}{dt} = -\left(\frac{B^2l^2}{Rm}\right)v$. Hence $v = v_0e^{-\frac{B^2l^2}{mR}t}$

(d) The energy goes into heat in the resistor. The power delivered to resistor is I^2R, so

$$\frac{dW}{dt} = I^2R = \frac{B^2l^2v^2}{R^2}R = \frac{B^2l^2}{R}v_0^2e^{-2\alpha t}$$

we have taken $\alpha \equiv \frac{B^2l^2}{mR}$; So $\frac{dW}{dt} = \alpha mv_0^2e^{-2\alpha t}$

The total energy delivered to the resistor is

$$W = \alpha mv_0^2\int_0^\infty e^{-2\alpha t}dt = \alpha mv_0^2\frac{e^{-2\alpha t}}{-2\alpha}\bigg|_0^\infty = \alpha mv_0^2\frac{1}{2\alpha} = \frac{1}{2}mv_0^2$$

Example: Faraday's disk generator: A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform field B , pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk (See figure). Find the current in the resistor.

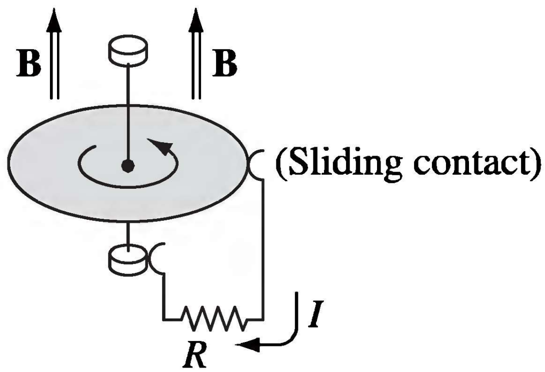

Figure 6.2: Current created by rotating metal disk

Solution: The speed of a point on the disk at a distance s from the axis is $v = \omega s$, so the force per unit charge is $\mathbf{f}_{\text{mag}} = \mathbf{v} \times \mathbf{B} = \omega s B \hat{\mathbf{s}}$. The emf is therefore

$$\mathcal{E} = \int_0^a f_{\text{mag}} ds = \omega B \int_0^a s ds = \frac{\omega B a^2}{2}$$

So the current is

$$I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}$$

The flux law or Faraday-Letz rule can also be written as in terms of electric field

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \tag{6.5}$$

Example: A uniform magnetic field $\mathbf{B}(t)$, pointing straight up, fills the shaded circular region, made by conducting material, as shown in the figure below. If $\mathbf{B}$ is changing with time, what is the induced electric field?

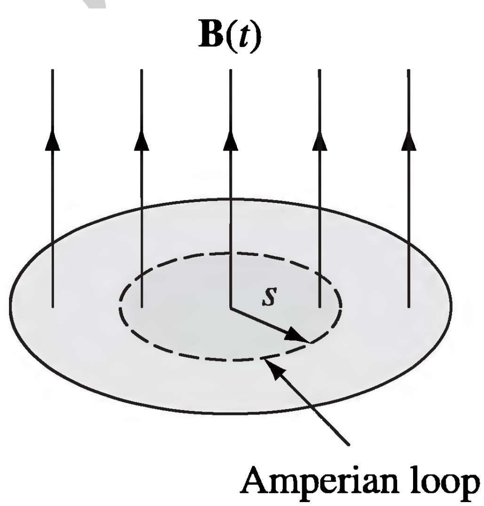

Figure 6.3: Induced Current by changing magnetic field

Solution: **E** points in the circumferential direction, just like the magnetic field inside a long straight wire carrying a uniform current density. Draw an Amperian loop of radius s, and apply Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2 \frac{dB}{dt}$$

Hence

$$\mathbf{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\boldsymbol{\phi}}$$

If **B** is increasing, **E** runs clockwise, as viewed from above.

Example: A line charge λ is glued onto the rim of a wheel of radius b,. The spokes are made of some nonconducting material. The wheel is then suspended horizontally, as shown in figure below so that it is free to rotate. In the central region, which is made by conducting material, out to radius a, there is a uniform magnetic field $\mathbf{B}_0$, pointing up. Now someone turns the field off. What happens?

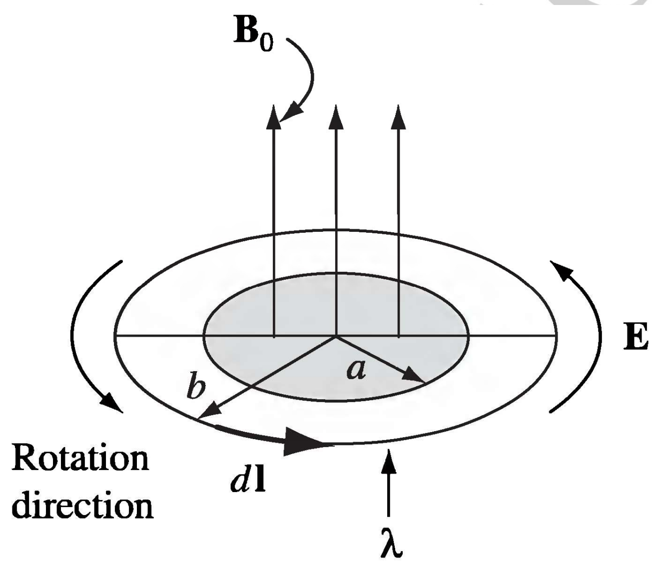

Figure 6.4: Charged disk rotates because of changing **B**

solution: The changing magnetic field will induce an electric field, curling around the axis of the wheel. This electric field exerts a force on the charges at the rim, and the wheel starts to turn. According to Lenz's law, it will rotate in such a direction that its field tends to restore the upward flux. The motion, then, is counterclockwise, as viewed from above.

Faraday's law, applied to the loop at radius b , says

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi b) = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}, \quad \text{or} \quad \mathbf{E} = -\frac{a^2}{2b}\frac{dB}{dt}\hat{\boldsymbol{\phi}}$$

The torque on a segment of length $d\mathbf{l}$ is $(\mathbf{r} \times \mathbf{F})$, or $b\lambda E$ dl. The total torque on the wheel is therefore

$$N = b\lambda\left(-\frac{a^2}{2b}\frac{dB}{dt}\right)\oint dl = -b\lambda\pi a^2\frac{dB}{dt}$$

The angular momentum imparted to the wheel is

$$\int N dt = -\lambda\pi a^2 b\int_{B_0}^{0} dB = \lambda\pi a^2 b B_0$$

It doesn't matter how quickly or slowly you turn off the field; the resulting angular velocity of the wheel is the same.

Now the question is - where is the angular momentum coming from? wait for the next section.